MA 161: Lesson 28

Linear Approximation and Differentials (4.6)

Linear approximation of a function

Using tangent line to approximate function values

Error and percentage error

Announcements

Exam 3 on Nov 20th - Lesson 18 (3.10) to Lesson 30 (4.9)

Study guide and instructions for exam 3 posted on brightspace

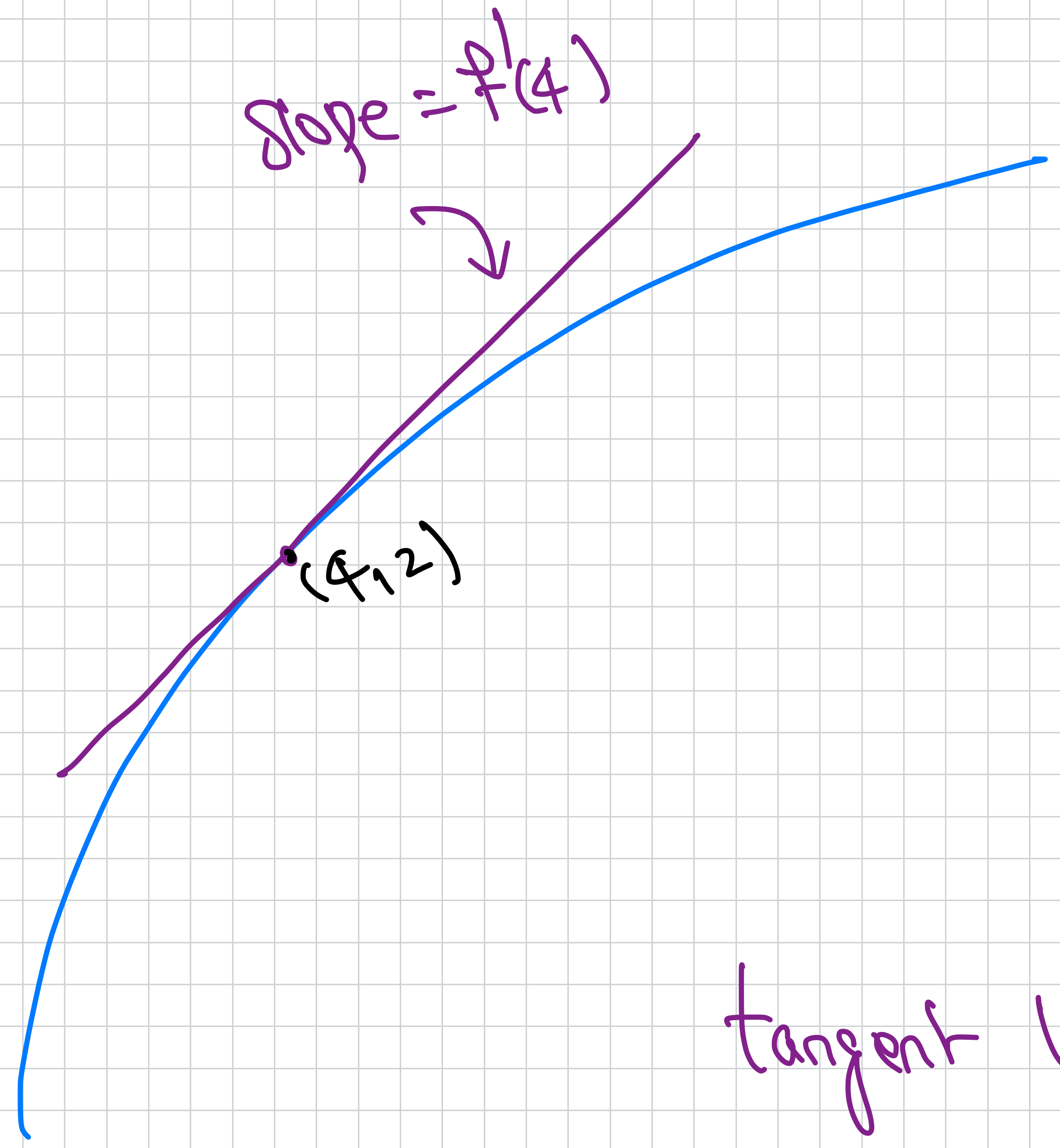
Feasting with faculty - tomorrow 11:15am -12:15pm (Windsor)

Office Hours

M, W, F: 245pm - 415pm

Warmup: Finding equation of a tangent line

find equation of tangent line to $f(x)=\sqrt{x}$ at $(4,2)$



slope of tangent line = Derivative

$$f'(x) = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2\sqrt{x}}$$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

tangent line $\rightsquigarrow$ line through $\underbrace{(4,2)}_{f(4)}$ with slope $1/4$.

$$y - f(4) = f'(4)(x-4) \rightsquigarrow \boxed{y = 2 + \frac{1}{4}(x-4)}$$

Linear Approximation

Approximating the function using the tangent line at a point

Linear Approx. of $f(x)$ at $x = a$ is

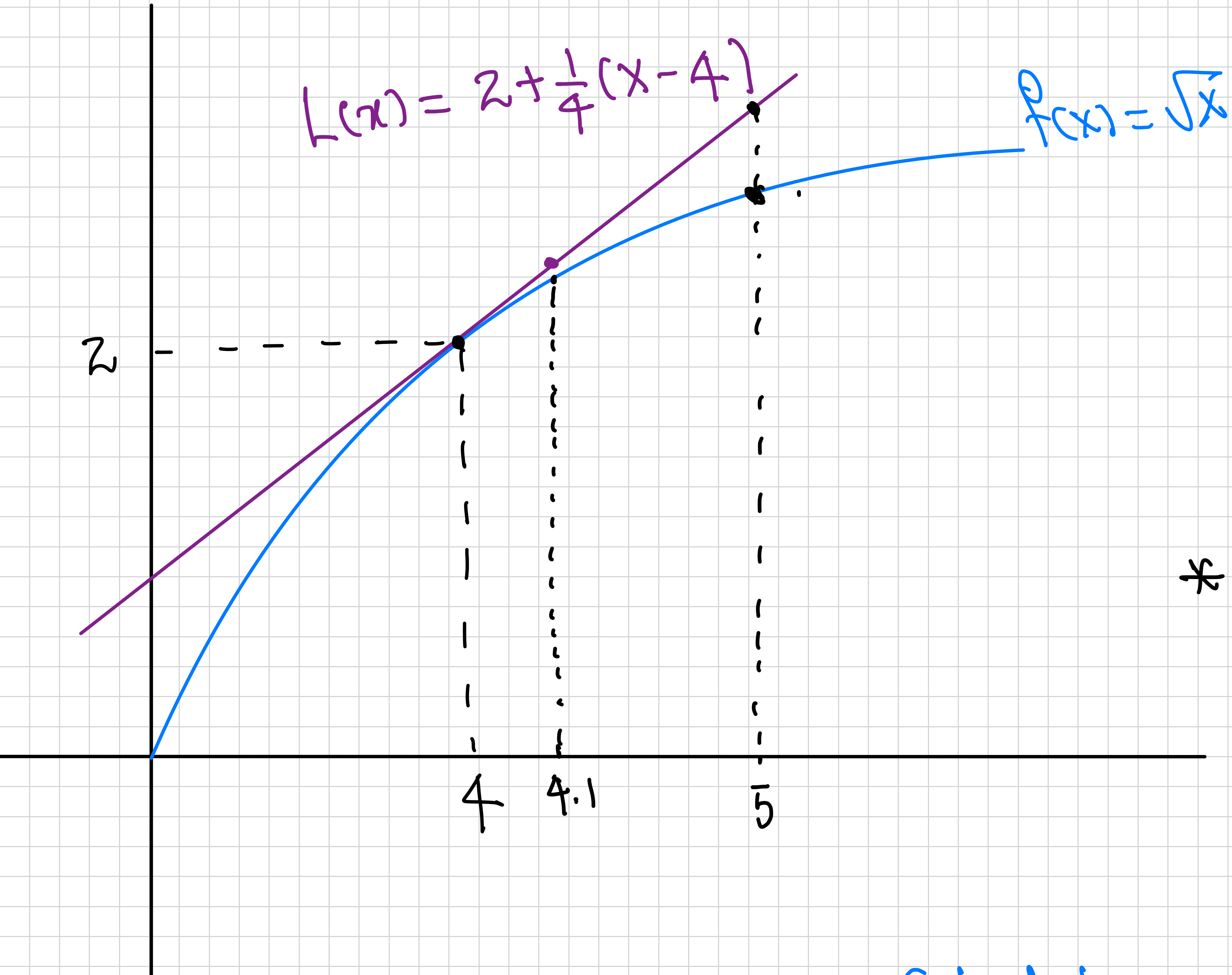
$$L(x) = f(a) + f'(a)(x-a)$$

called Differential.

When x is close to a, we can Approximate $f(x)$ with value of $L(x)$.

$$\text{ERROR} = |\text{Actual} - \text{Approximate}|$$

$$\%\ \text{ERROR} = 100 \times \frac{|\text{Actual} - \text{Approximate}|}{|\text{Actual}|}$$



We Can Approximate

$\sqrt{4.1}, \sqrt{5}, \ldots$

with the help of Straight line

* $f(4) = 2$

  $L(4) = 2$

calculator   Actual

* $f(4.1) \approx 2.0248$

Approx

  $L(4.1) = 2 + \frac{1}{4}(4.1 - 4) = 2.025$

  $|\text{Actual} - \text{Approx}| = 0.0002.$

Calculator

* $f(5) = \sqrt{5} \approx 2.236$

  $L(5) = 2 + \frac{1}{4}(5-4) = 2.25$

  $|\text{Actual} - \text{Approx}| = 0.\underline{014}$

eg: Use Linear Approximation to estimate the value of $\sqrt{40}$

$f(x) = \sqrt{x}$, linear Approx at $x =$ a number close to 40 whose exact function you know.

36.

$L(x) = \underbrace{f(36)}_{\sqrt{36} = 6} + \underbrace{f'(36)}_{f'(x) = \frac{1}{2\sqrt{x}}, \; f'(36) = \frac{1}{12}} (x - 36)$

$$\boxed{L(x) = 6 + \frac{1}{12}(x - 36)}$$

$$\sqrt{40} = f(40) \approx L(40) = 6 + \frac{1}{12}(40 - 36) = 6.333$$

Eg: Use Linear Approximation to estimate the value of $\ln(1.2)$

$f(x) = \ln x$

Approx at $x = 1$

$L(x) = f(1) + f'(1)\,(x-1)$

$\ln 1 = 0$

$f'(x) = \frac{1}{x}$

$f'(1) = 1$

$L(x) = x - 1$

is L.A of $\ln x$ at $x = 1$

$\ln(1.2) = f(1.2) \approx L(1.2) = 1.2 - 1 = \underline{\underline{0.2}}$

eg: Find Linear approximation of $f(x) = \sin x$ at $x = 0$

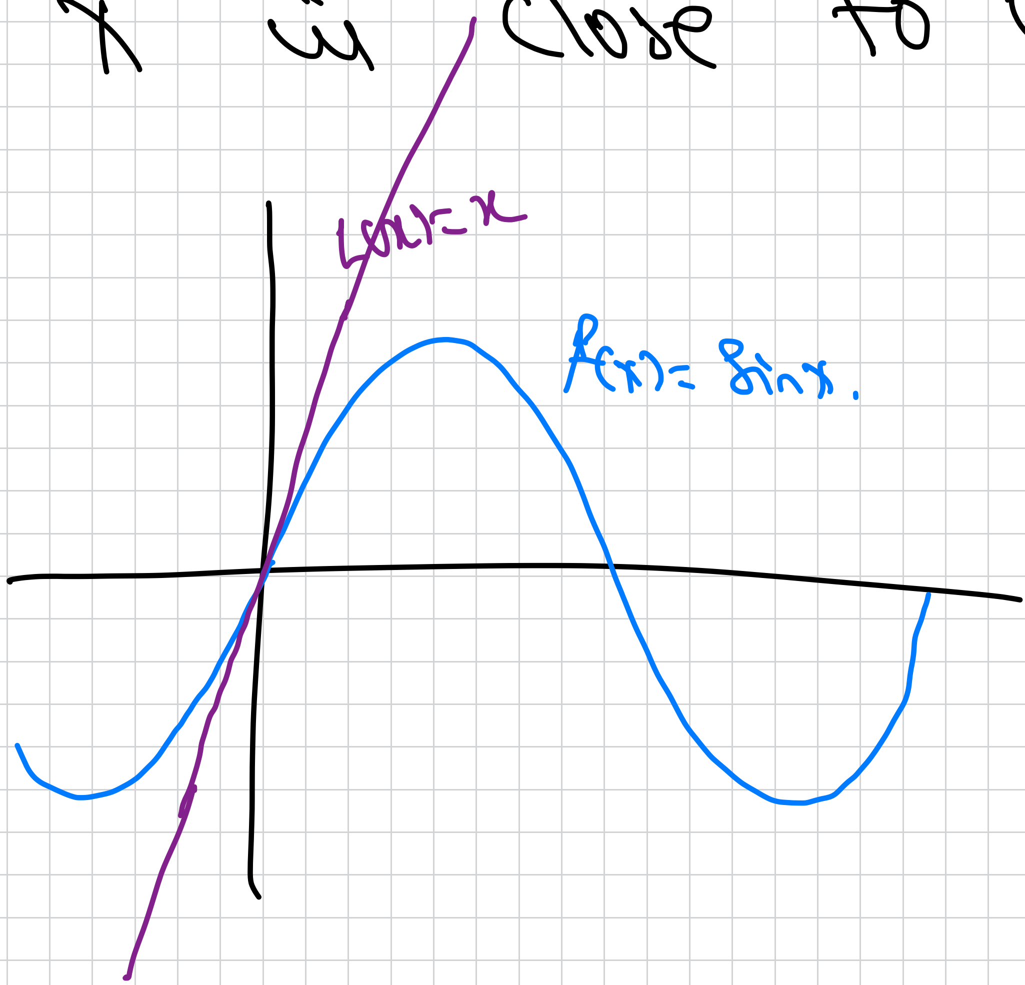
$$L(x) = f(0) + f'(0)(x-0) \rightsquigarrow \boxed{L(x) = x}$$

$f(0) = \sin(0) = 0$

$f'(x) = \cos x$

$f'(0) = \cos(0) = 1$

When x is close to 0 $\sin x \approx L(x) = x$.



$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

$$\sin\left(\frac{\pi}{20}\right) \approx L\left(\frac{\pi}{20}\right) = \frac{\pi}{20}$$

Use linear approximation of $f(x) = \sin x$ at $x=0$ to approximate values of $\sin(\pi/6)$, $\sin(\pi/3)$, $\sin(\pi/2)$

$= \frac{1}{2} = 0.5$ $\frac{\sqrt{3}}{2} = 0.866$ 1

$L(x) = x$

① $L(\pi/6) = \pi/6 = 0.52$ $\leadsto$ $\text{ERROR} = \left| f(\pi/6) - L(\pi/6) \right| = 0.02$

$\% \text{ ERROR} = 100 * \dfrac{\text{ERROR}}{|f(\pi/6)|} = 4\%$

② $L(\pi/3) = \pi/3 \approx 1.04$ $\leadsto$ $\text{ERROR} = |0.866 - 1.04|$

$\% \text{ ERROR} = 100 * \dfrac{\text{ERROR}}{|0.866|} \approx 20\%$

③ $L(\pi/2) = \pi/2 \approx 1.57$ $\leadsto$ $\% \text{ ERROR} = 100 * \dfrac{|1 - 1.57|}{|1|} \approx 57\%$

e.g.: $f(x) = 8 - 3x^2$

Use Linear Approximation at $x = 2$ to estimate $f(1.9)$

$L(x) = f(2) + f'(2)(x-2)$

$f(2) = 8 - 3(2)^2$
$= -4$

$f'(x) = -6x$
$f'(2) = -12$

$L(x) = -4 - 12(x-2)$

$f(1.9) \approx L(1.9) = -4 - 12(1.9 - 2) = \underline{\underline{-2.8}}$.

$= 8 - 3(1.9)^2$
$= 8 - 3(3.61)$
$= \underline{-2.83}$

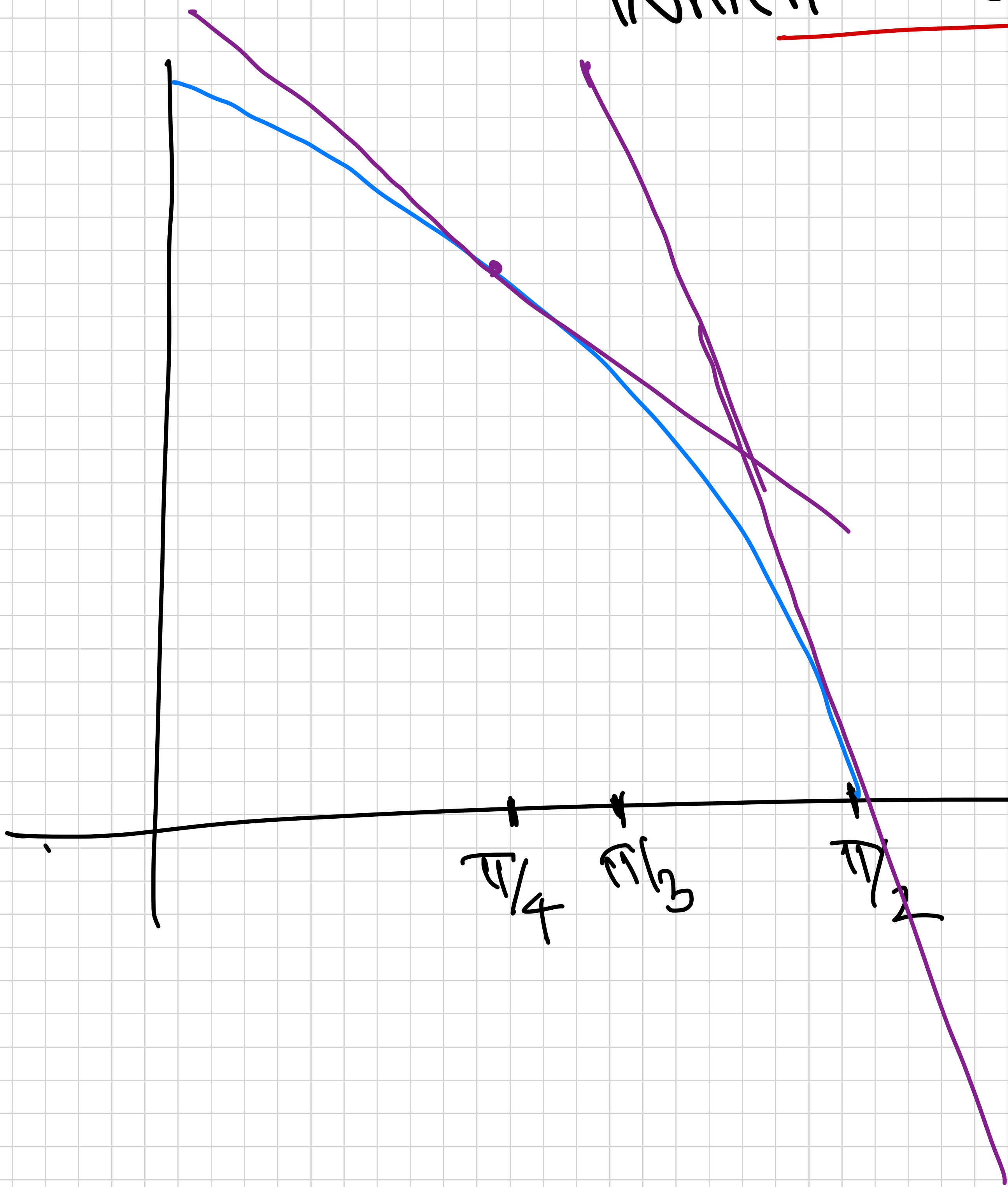
$\text{ERROR} = |\text{Actual} - \text{Approx}| = |-2.83 + 2.8|$
$= \underline{\underline{0.03}}$

$\% \text{ ERROR} = 100 \times \frac{\text{ERROR}}{|\text{Actual}|} \approx \underline{\underline{1\%}}$

eg: Estimate the value of $\cos(\pi/3)$, using linear Approx of $f(x) = \cos x$ at $x = \pi/4$ and at $x = \pi/2$

Which one is a Better approximate?

Guess: $\pi/4$



Linear Approx at $x = \pi/4$

$$L_1(x) = f(\pi/4) + f'(\pi/4)(x - \pi/4)$$

$$L_1(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \pi/4\right)$$

Linear Approx. at $\pi/2$

$$L_2(x) = f(\pi/2) + f'(\pi/2)(x - \pi/2)$$

$$L_2(x) = -(x - \pi/2) = \pi/2 - x.$$

$f(x) = \cos x \rightsquigarrow f(\pi/3) = \cos \pi/3 = 1/2 = 0.5$

$L_1(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \pi/4\right)$ $\rightsquigarrow$ Approx. at $\pi/4$

$$L_1(\pi/3) = \frac{1}{\sqrt{2}}\left[1 - \frac{\pi}{3} + \frac{\pi}{4}\right] = \frac{1}{\sqrt{2}}\left[\frac{12 - \pi}{12}\right] \approx 0.521$$

$L_2(x) = \frac{\pi}{2} - x$ $\rightsquigarrow$ Approx at $\pi/2$

$$L_2(\pi/3) = \frac{\pi}{2} - \frac{\pi}{3} = \pi/6 \approx 0.5235$$

L.A at $\frac{\pi}{4}$ is Better.